Note on a product formula for unitary groups

by Vincent CACHIA[1]

Abstract: For any nonnegative self-adjoint operators A and B in a separable Hilbert space, we show that the Trotter-type formula $[(e^{i2tA/n}+e^{i2tB/n})/2]^n$ converges strongly in $\mathrm{dom}(A^{1/2})\cap\mathrm{dom}(B^{1/2})$ for some subsequence and for almost every $t\in\mathbb{R}$. This result extends to the degenerate case and to Kato-functions following the method of Kato [6].

In a famous paper [6], T. Kato proved that for any nonnegative self-adjoint operators A and B in a Hilbert space $\mathcal{H}$, the Trotter product formula $(e^{-tA/n}e^{-tB/n})^n$ converges strongly to the (degenerate) semigroup generated by the form-sum $A\dot{+}B$ for any t with $\mathrm{Re}\,t>0$. He also extended the result to a class of so-called Kato-functions, and to degenerate semigroups. However the convergence on the boundary $i\mathbb{R}$ remains an unclear problem in this generality [1, 3]. For Kato-functions f such that $\mathrm{Im}\,f\leq 0$ (for example $f(s)=(1+is)^{-1}$), Lapidus found such an extension [7]. For the case of the Trotter product formula with projector $(e^{itA/n}P)^n$, Exner and Ichinose obtained recently a interesting result [4].

1 Statement of the result

Since this note is closely related to Kato's paper [6], it is convenient to use similar notations. A and B denote nonnegative self-adjoint operators defined in closed subspaces M_A and M_B of a separable Hilbert space $\mathcal{H}$, and P_A, P_B denote the orthogonal projections on M_A and M_B. Let $\mathcal{D}'=\mathrm{dom}(A^{1/2})\cap\mathrm{dom}(B^{1/2})$, let $\mathcal{H}'$ be the closure of $\mathcal{D}'$, and let P' be the orthogonal projector on $\mathcal{H}'$. The form-sum $C=A\dot{+}B$ is defined as the self-adjoint operator in $\mathcal{H}'$ associated with the nonnegative, closed quadratic form $u\mapsto\|A^{1/2}u\|^2+\|B^{1/2}u\|^2$, $u\in\mathcal{D}'$. We consider Trotter-type product formulae $F(t/n)^n$ based on the arithmetic mean

$$F(t)=\frac{f(2tA)P_A+g(2tB)P_B}{2}. \tag{1}$$

The Kato-functions f and g are assumed here to be bounded, holomorphic functions in $\{t\in\mathbb{C}:\mathrm{Re}\,t>0\}$ with:

$$|f(t)|\leq 1,\ f(0)=1,\ f'(+0)=\lim_{t\to 0,\mathrm{Re}\,t>0}\frac{f(t)-1}{t}=-1, \tag{2}$$

and $0\leq f(s)\leq 1$ if $s>0$, and the same conditions for g. By the functional calculus for normal operators, $F(t)$ is well defined for $\mathrm{Re}\,t\geq 0$ and bounded by 1.

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Theorem 1 *Let $\mathcal{H}$ be a separable Hilbert space. Let A, M_A, P_A, B, M_B, P_B, C, $\mathcal{H}'$, P', f, g, and F be as defined above. For any $u \in \mathcal{H}$ one has*

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \phi(t) F(it/n)^n u dt = \int_{-\infty}^{+\infty} \phi(t) e^{-itC} P' u dt, \ \phi \in L^1(\mathbb{R}). \tag{3}$$

Moreover there exists a set $L \subset \mathbb{R}$ with zero Lebesgue measure and an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, such that:

$$\forall u \in \mathcal{H}', \quad F(it/\varphi(n))^{\varphi(n)} u \longrightarrow e^{-itC} P' u, \ t \in \mathbb{R} \setminus L, \tag{4}$$

whereas for $u \in M_A^\perp + M_B^\perp$, $\lim_{n\to\infty} F(it/n)^n u = 0$, for each $t \in \mathbb{R}$.

One sees that the strong convergence, valid in the open right half-plane, cannot extend exactly to the boundary $i\mathbb{R}$, as already remarked in [3, 4]: the strong convergence on the boundary is restricted to the subspace $\mathcal{H}' + M_A^\perp + M_B^\perp$. However the weaker convergence (3) was already observed [1, 5].

2 Proof

Let us consider for $\mathrm{Re}\, t \geq 0$ and $\tau > 0$:

$$S_{t,\tau} = \tau^{-1}(I - F(t\tau)) \tag{5}$$

which is a holomorphic operator-valued function of t in the open right half-plane.

The main step of the proof is to show that the strong convergence:

$$s - \lim_{\tau\to 0} (I + S_{t,\tau})^{-1} = (I + tC)^{-1} P' \tag{6}$$

holds for $\mathrm{Re}\, t > 0$, and remains true for almost all $t \in i\mathbb{R}$ and on some subsequence. This will give the announced result (4) for $u \in \mathcal{H}'$ by Chernoff's lemma (see below). The convergence on the boundary is obtained by a useful result of Feldman [5, Th 5.1], which we state here in a slightly more general form:

Lemma 2 *Let $\{\Psi_\tau : 0 < \tau < 1\}$ be a uniformly bounded family of bounded holomorphic $\mathcal{H}$-valued function defined in the open right half-plane. Suppose that $\Psi_\tau(z) \xrightarrow{\tau\to 0} \Psi(z)$, for each z with $\mathrm{Re}\, z > 0$. Then for each $v \in L^1(\mathbb{R}, \mathcal{H})$*

$$\int_{\mathbb{R}} (v(t), \Psi_\tau(it)) dt \xrightarrow{\tau\to 0} \int_{\mathbb{R}} (v(t), \Psi(it)) dt. \tag{7}$$

Moreover for each (numerical) $\phi \in L^1(\mathbb{R})$:

$$s - \lim_{\tau\to 0} \int_{\mathbb{R}} \phi(t) \Psi_\tau(it) dt = \int_{\mathbb{R}} \phi(t) \Psi(it) dt. \tag{8}$$

Proof: The two results (7) and(8) follow from very similar arguments, so we present only the first one. The bounded holomorphic functions Ψ_τ have boundary values for almost every $is \in i\mathbb{R}$, and for any $t > 0$ and $s \in \mathbb{R}$,

$$\Psi_\tau(t + is) = \int_{-\infty}^{+\infty} \frac{t/\pi}{t^2 + (s - s')^2} \Psi_\tau(is') ds' = P_t * \Psi_\tau(i\cdot).$$

The kernel P_t is in fact an approximate identity: $P_t * \phi \xrightarrow{L^1} \phi$ as $t \to 0$. Then we have:

$$\begin{aligned}\int_{\mathbb{R}} (v(s), [\Psi_\tau(is) - \Psi(is)])ds &= \int_{\mathbb{R}} (v(s), [\Psi_\tau(t+is) - \Psi(t+is)])ds + \\ &+ \int_{\mathbb{R}} ([v(s) - (P_t * v)(s)], [\Psi_\tau(is) - \Psi(is)])ds,\end{aligned}$$

where we have used:

$$\int_{\mathbb{R}} (v(s), [P_t * h](s))ds = \int_{\mathbb{R}} ([P_t * v](s), h(s))ds.$$

This leads to

$$\left| \int_{\mathbb{R}} (v(s), [\Psi_\tau(is) - \Psi(is)])ds \right| \leq \int_{\mathbb{R}} \|v(s)\| \|\Psi_\tau(t+is) - \Psi(t+is)\| ds \tag{9}$$

$$+ \int_{\mathbb{R}} \|v(s) - (P_t * v)(s)\| \|\Psi_\tau(is) - \Psi(is)\| ds. \tag{10}$$

The last term (10) can be made arbitrary small by choosing t sufficiently small, and for any t the integral in the right hand side of (9) tends to 0 as $\tau \to 0$ by Lebesgue's theorem.

□

Remark: It can also be shown that (8) implies (7).

It is convenient to introduce the following bounded accretive operators, for $\mathrm{Re}\, t \geq 0$ and $\tau > 0$:

$$A_{t,\tau} = \tau^{-1}[I - f(t\tau A)P_A], \text{ and } B_{t,\tau} = \tau^{-1}[I - g(t\tau B)P_B]. \tag{11}$$

Lemma 3 *For any t with $\mathrm{Re}\, t > 0$, $s-\lim_{\tau\to 0}(I+S_{t,\tau})^{-1} = (I+tC)^{-1}P'$. Moreover for any $v \in L^1(\mathbb{R}, \mathcal{H})$, $u \in \mathcal{H}$ and $t \in \mathbb{R}$, one has*

$$\lim_{\tau\to 0} \int_{\mathbb{R}} (v(t), (I + S_{it,\tau})^{-1}u)dt = \int_{\mathbb{R}} (v(t), (I + itC)^{-1}P'u)dt. \tag{12}$$

Proof: Since $S_{t,\tau} = A_{t,2\tau} + B_{t,2\tau}$, the strong convergence of $(I + S_{t,\tau})^{-1}$ for $t > 0$ follows from [6, Lem. 2.2 and 2.3]. Then it extends to the open right half-plane by the theorem of Vitali: for any $\tau > 0$, $(I + S_{t,\tau})^{-1}$ is a holomorphic function of t, and is bounded by 1.

The convergence on the boundary (12) follows from Lemma 2. □

For any fixed $u \in \mathcal{H}$ and $t \in \mathbb{R}$ we set $w_{t,\tau} = (I + S_{it,\tau})^{-1}u$, $\tau > 0$. Then one finds

$$(u, w_{t,\tau}) = \|w_{t,\tau}\|^2 + (A_{it,2\tau}w_{t,\tau}, w_{t,\tau}) + (B_{it,2\tau}w_{t,\tau}, w_{t,\tau}) \tag{13}$$

with $\mathrm{Re}\,(A_{it,2\tau}w_{t,\tau}, w_{t,\tau}) \geq 0$ and $\mathrm{Re}\,(B_{it,2\tau}w_{t,\tau}, w_{t,\tau}) \geq 0$. Therefore

$$\|w_{t,\tau}\|^2 \leq \mathrm{Re}\,(u, w_{t,\tau}) \leq |(u, w_{t,\tau})| \leq \|u\|\|w_{t,\tau}\| \tag{14}$$

and thus $\|w_{t,\tau}\| \leq \|u\|$, $\tau > 0$.

Lemma 4 *Let α_n be any sequence of positive numbers with limit zero. There exists a set $L \subset \mathbb{R}$ of zero Lebesgue measure and a subsequence τ_n of α_n, such that for each $t \in \mathbb{R} \setminus L$, $s-\lim_{n\to\infty}(I + S_{it,\tau_n})^{-1} = (I + itC)^{-1}P'$.*

Proof: It follows from Lemma 3 that: $\int_{\mathbb{R}}(1+t^2)^{-1}(u, w_{t,\tau})dt \to \int_{\mathbb{R}}(1+t^2)^{-1}(u, w_t)dt$ with $w_t = (I + itC)^{-1}P'u$. Thus the same is true for the real part, and we have by (13):

$$\mathrm{Re}\,(u, w_{t,\tau}) = \|w_{t,\tau}\|^2 + \|(\mathrm{Re}\,A_{it,2\tau})^{1/2}w_{t,\tau}\|^2 + \|(\mathrm{Re}\,B_{it,2\tau})^{1/2}w_{t,\tau}\|^2$$

We observe that $\mathrm{Re}\,(u, w_t) = \mathrm{Re}\,((I + itC)w_t, w_t) = \|w_t\|^2$, and that

$$\int_{\mathbb{R}} \mathrm{Re}\,(w_{t,\tau}, w_t)\frac{dt}{1+t^2} \xrightarrow{\tau\to 0} \int_{\mathbb{R}} \|w_t\|^2\frac{dt}{1+t^2}.$$

Then one finds

$$\int_{\mathbb{R}} \left(\|w_{t,\tau} - w_t\|^2 + \|(\mathrm{Re}\,A_{it,2\tau})^{1/2}w_{t,\tau}\|^2 + \|(\mathrm{Re}\,B_{it,2\tau})^{1/2}w_{t,\tau}\|^2\right)\frac{dt}{1+t^2} \xrightarrow{\tau\to 0} 0,$$

in particular $\int_{\mathbb{R}} \|w_{t,\tau} - w_t\|^2(1 + t^2)^{-1}dt \xrightarrow{\tau\to 0} 0$. This means that the functions $t \mapsto \|w_{t,\tau} - w_t\|$ converge to 0 in $L^2(\mathbb{R}, \mu)$ as $\tau \to 0$, with the finite measure $d\mu = (1 + t^2)^{-1}dt$. Let $(e_m)_{m\in\mathbb{N}}$ be a basis of the separable Hilbert space $\mathcal{H}$. For $u = e_1$ the above L^2-convergence implies that there exists $L_1 \subset \mathbb{R}$ with $\mu(L_1) = 0$ and some increasing function $\varphi_1 : \mathbb{N} \to \mathbb{N}$ such that $(I + S_{it,\alpha_{\varphi_1(n)}})^{-1}e_1 \to (I + itC)^{-1}P'e_1$ as $n \to \infty$, for any $t \in \mathbb{R}\setminus L_1$. Then for $u = e_2$ there exists $L_2 \subset \mathbb{R}$ with $\mu(L_2) = 0$ and an increasing function $\varphi_2 : \mathbb{N} \to \mathbb{N}$, such that $(I+S_{it,\alpha_{\varphi_1\circ\varphi_2(n)}})^{-1}e_2 \to (I+itC)^{-1}P'e_2$ as $n \to \infty$, for any $t \in \mathbb{R} \setminus L_2$, and so on for each $m \in \mathbb{N}$. Finally by the diagonal procedure we consider the sequence $\tau_n = \alpha_{\varphi_1\circ\cdots\circ\varphi_n(n)}$ and find that convergence holds for each vector e_m of the basis and for each $t \in \mathbb{R} \setminus L$, where $L = \cup_{m\in\mathbb{N}}L_m$. We have $\mu(L) = 0$ and thus L has also zero Lebesgue measure. Since the operators $(I + S_{it,\tau})^{-1}$ are uniformly bounded, this implies the strong convergence for any vector $u \in \mathcal{H}$.

□

Proof of the theorem: We consider $Z_{t,n} = (n/t)[F(it/n) - I] = -t^{-1}S_{it,1/n}$ and $\alpha_n = 1/n$. Let L be as in Lemma 4 and let $t \in \mathbb{R} \setminus L$, $t \neq 0$. By [2, Th. 3.17] and Lemma 4 one obtains for some increasing function $\varphi : \mathbb{N} \to \mathbb{N}$:

$$\lim_{n\to\infty} e^{sZ_{t,\varphi(n)}}u = e^{-isC}P'u, \quad u \in \mathcal{H}',\ s \in \mathbb{R}. \tag{15}$$

By Chernoff's Lemma [2, Lem. 3.27 and 3.29], one has

$$\lim_{n\to\infty} \|F(it/\varphi(n))^{\varphi(n)}u - e^{\varphi(n)(F(it/\varphi(n))-I)}u\| = 0, \quad u \in \mathcal{H}'. \tag{16}$$

Thus we obtain the convergence (4) in $\mathcal{H}'$. If $u \in M_A^\perp + M_B^\perp$, the convergence to 0 is clear because $F(it/n)^n$ reduces to $f(2it/n)^nP_A/2^n$ or $g(2it/n)^nP_B/2^n$. The $\sigma(L^\infty, L^1)$ convergence (3) follows by using Kato's result for $\mathrm{Re}\,t > 0$ and Lemma 2.

□

Remark: the subsequence appearing in the theorem makes the result somewhat unsatisfactory. In fact this restriction is not necessary if we assume that the functions $t \mapsto (I + S_{it,\tau})^{-1}u$ are equicontinuous with respect to τ, at some point $t_0 \neq 0$. In this case Lemma 4 can be improved in the following way:

$$s - \lim_{\tau \to 0}(I + S_{it_0,\tau})^{-1} = (I + it_0 C)^{-1}P'. \tag{17}$$

For the proof, let us consider an approximate identity $\rho_n : \mathbb{R} \mapsto \mathbb{R}_+$. By Lemma 3 one has $\lim_{\tau\to 0}[\rho_n * (u, w_{\cdot,\tau})](t_0) = [\rho_n * (u, w_{\cdot})](t_0)$ for each $n = 1, 2, \ldots$, and by the equicontinuity of the functions $t \mapsto (u, w_{t,\tau})$ at t_0, $\lim_{n\to\infty}[\rho_n * (u, w_{\cdot,\tau})](t_0) = (u, w_{t_0,\tau})$ uniformly in τ. Then in the proof of the theorem we consider $Z_{t,n} = -t_0^{-1} S_{it_0,t/nt_0}$ for any $t \in \mathbb{R}$. By (17) one has $s - \lim_{n\to\infty}(t_0^{-1} - Z_{t,n})^{-1} = (t_0^{-1} + iC)^{-1}P'$, which leads to the result of the theorem without subsequence (the exceptional set L has also disappeared). Concerning the equicontinuity condition, we recall that in our first result the function $t \mapsto F(it)$ $(t \in \mathbb{R})$ is not necessarily continuous (whereas it is continuous for example if f and g are the exponential function).

References

[1] Chernoff P.R.: Product Formulas, Nonlinear Semigroups, and Addition of Unbounded Operators, *Mem. Amer. Math. Soc.* **140**, Providence R.I. 1974.

[2] Davies E.B.: *One parameter semigroups*, Academic Press, London 1980.

[3] Exner P.: *Open Quantum Systems amd Feynman Integrals*, D. Reidel Publ. Co., Dordrecht 1985.

[4] Exner P., Ichinose T.:Product formula related to quantum Zeno dynamics, preprint 2003.

[5] Feldman J.: On the Schrödinger and heat equations for nonnegative potentials, *Trans. Amer. Math. Soc.* **108** (1963) 251-264.

[6] Kato T.: Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroup, in *Topics in functional Analysis (M. Kac ed.)*, Academic Press, New-York 1978, pp.185-195.

[7] Lapidus M.: Formule de moyenne et de produit pour les résolvantes imaginaires d'opérateurs auto-adjoints, *C.R. Acad. Sc. Paris série A*, **291** (1980) 451-454.